The proof rules for Identity

The proof rules $\forall$I, $\forall$E, $\exists$I, and $\exists$E together with the primitive proof rules of SL form a complete set of rules for any inferences that do not involve the identity symbol. However, if the language contains that symbol (as ours does) we need two additional rules to for a complete set – =I and =E.

The identity introduction (=I) rule is simple. You can simply write any identity statement of the form $\alpha=\alpha$ any time you feel like it depending on no assumptions at all. In other words, you can write a=a, b=b, c=c, etc. This rule allows you to prove a few theorems about identity such as.

(1) $a=a$	=I	(1) $a=a$	=I
(2) $\forall x\ x=x$	1 $\forall$I	(2) $\exists y\ a=y$	1 $\exists$I
		(3) $\forall x \exists y\ x=y$	2 $\forall$I

The proof on the left shows that everything is equal to itself and the proof on the right shows us that for anything at all, it is equal to something (namely, itself.) There are no assumptions written to the left of any line because none of the lines depend on any assumptions. The =I rule is not used very often in interesting proofs but it is needed to form a complete rule system.

The =E rule is much more interesting. It represents what is generally known in philosophy as Leibniz's Law. This law states that if a and b stand for the same thing, then anything true of a must be true of b. For example, given that we have a=b, if Pa is true, then Pb must be true. If ~Pa is true, then ~Pb must be true. If $\forall x \exists y(Rxy \ \&\ Rxa)$ is true then $\forall x \exists y(Rxy \ \&\ Rxb)$ must be true. The relaxed version of the rule (which we use) allows each of these inferences from a=b or from b=a. In other words, from Pa together with either a=b or b=a we can get Pb.

It is very important to note the correct direction of this rule. From the sentence a=b and a statement containing a, you can derive a statement containing b. You <u>cannot</u> derive a=b just from two sentences that are alike. For example, from Pa and Pb it is <u>not</u> correct to infer a=b. Maybe Adam is a painter and Bob is a painter. This does not mean that Adam is Bob. However, if 'Adam' and 'Bob' referring to the same person (maybe one is a nickname) then if Adam was a painter, Bob would be too.

Here are some example proofs:

EXAMPLE 1: $\forall x(Px \rightarrow x=a), \exists x(Px \ \&\ Qx) \vdash Qa$

Step 1. My second premise is an existential so I will make up a name for it so I can use $\exists$E. The name needs to be arbitrary so I can't use 'a'. But any other name will do.	1	(1) $\forall x(Px \rightarrow x=a)$	A
	2	(2) $\exists x(Px \ \&\ Qx)$	A
	3	(3) $Pb \ \&\ Qb$	A

	(n-1) Qa	new goal
	(n) Qa	∃E

Step 2. Now to use premise 1, I will plug ‘b’ in for ‘x’. Once I do that, it is easy to get b=a. But now that I know b=a, anything that is true of b must also be true of a. In particular, since Qb is true, Qa is also true.

1	(1) ∀x(Px → x=a)	A
2	(2) ∃x(Px & Qx)	A
3	(3) Pb & Qb	A
3	(4) Pb	3 &E
3	(5) Qb	3 &E
1	(6) Pb → b=a	1 ∀E
1,3	(7) b=a	4,6 →E
1,3	(8) Qa	5,7 =E
1,2	(9) Qa	2,8 ∃E(3)

EXAMPLE 2: ∃x∀y x=y, ∀xRxx ├ ∀x∀yRxy

Step 1: Since my first premise is existential, I will make up a name for that thing so I can use ∃E. I will choose ‘a’ since it doesn’t matter. Now my goal is a universal so I will try to prove an arbitrary instance of it so I can use ∀I. Since this instance must be arbitrary it cannot contain ‘a’.

1	(1) ∃x∀y x=y	A
2	(2) ∀xRxx	A
3	(3) ∀y a=y	A
	Rbc	new goal
	(n-2) ∀yRby	∀I
	(n-1) ∀x∀yRxy	∀I
	(n) ∀x∀yRxy	∃E

Step 2: There are many different ways of finishing this proof. They all involve realizing that we need to get an ‘R’ sentence from 2 and then use =E to manipulate it to get Rbc. I will plug in ‘a’ to line 2 and then replace the ‘a’s with b and c. It would also work to get, say, Rbb and then replace the second ‘b’ with ‘c’. Of course to do that we would need b=c which we could get from a=b and a=c.

1	(1) ∃x∀y x=y	A
2	(2) ∀xRxx	A
3	(3) ∀y a=y	A
2	(4) Raa	2 ∀E
3	(5) a=b	3 ∀E
2,3	(6) Rba	4,5 =E
3	(7) a=c	3 ∀E
2,3	(8) Rbc	6,7 =E
2,3	(9) ∀yRby	8 ∀I
2,3	(10) ∀x∀yRxy	9 ∀I
1,2	(11) ∀x∀yRxy	1,10 ∃E(3)

If we look carefully at the first line and the way in which it was used, it becomes clear that line 1 is a way of saying that there is only one thing in the universe. No matter what letter we come up with (say b or c) it is going to be equal to that first thing. Now, if there is only one thing and everything is related to itself (∀xRxx) then that one thing is related to itself (Raa). But this is everything there is so everything is related to everything (∀x∀yRxy).

The NI rule:

The NI rule (for negated identity) is a shortcut rule that gives us slightly more efficient ways of proving sentences of the form x≠y. It is important to think of this as the negation of x=y. For example, from x=y → Pa and ~Pa, we can derive x≠y by MT. The contraposition of Leibniz's Law allows us to infer that if two things do not share all of the same properties they must not be identical. In other words, from Pa and ~Pb we should be able to infer a≠b. The general form is that from one sentence containing α and its exact negation except for containing β instead of α, you can infer α≠β. This is fairly easily done via a reductio argument.

1	(1) Pa	A	1	(1) ∀x(Px→Pc)	A
2	(2) ~Pb	A	2	(2) ~∀x(Px→Pd)	A
3	(3) a=b	A	3	(3) c=d	A
2,3	(4) ~Pa	2,3 =E	2,3	(4) ∀x(Px→Pd)	1,3 =E
1,2	(5) a≠b	1,4 RAA(3)	1,2	(5) c≠d	1,4 RAA(3)

However, because this form of reasoning is clearly valid and so common in difficult proofs, I will allow you to skip the RAA reasoning and instead infer directly that α≠β by the NI rule:

1	(1) Pa	A	1	(1) ∀x(Px→Pc)	A
2	(2) ~Pb	A	2	(2) ~∀x(Px→Pd)	A
1,2	(3) a≠b	1,2 NI	1,2	(3) c≠d	1,2 NI

Note that it is important that one of the sentences be the exact negation of other except for the name change. For example, it is not correct to infer a≠b from Pa v Qc and ~Pb v Qc. Here only part of the sentence is the negation of a part of the other sentence. a=b is consistent with each of these sentences – for example, Qc may be true and thus make both sentences true.

EXAMPLE 3: ∀x(Gx → Hx), ∀x(Fx → Gx), Fa & ~Hb ⊢ a≠b

Step 1: My goal is a negated identity claim. I will try to get two sentences such that one is the exact contradictory of the other except that one contains 'a' where the other has 'b'.

1	(1) ∀x(Gx → Hx)	A
2	(2) ∀x(Fx → Gx)	A
3	(3) Fa & ~Hb	A
	(n) a≠b	NI

Step 2: This is fairly easy here. Simply plugging in an obvious letter (either a or b will work) to 1 and 2 will lead us to our goal sentences.

In this case, since I have Ha and ~Hb a and b must not be identical.

1	(1) ∀x(Gx → Hx)	A
2	(2) ∀x(Fx → Gx)	A
3	(3) Fa & ~Hb	A
3	(4) Fa	3 &E
3	(5) ~Hb	3 &E
1	(6) Ga → Ha	1 ∀E
2	(7) Fa → Ga	2 ∀E
2,3	(8) Ga	4,7 →E
1,2,3	(9) Ha	6,8 →E
1,2,3	(10) a≠b	5,9 NI

EXAMPLE 4: ∃x(Px & ∀y(~Rxy → x=y)) ├ ∀x(~Px → ∃y(y≠x & Ryx))

Step 1: My premise is existential so I introduce a name for it. My goal is universal so I will try to prove an arbitrary instance of it. Since my new goal is conditional, I assume its antecedent and try to prove its consequent.

1	(1) ∃x(Px & ∀y(~Rxy → x=y))	A
2	(2) Pa & ∀y(~Ray → a=y)	A
2	(3) Pa	2 &E
2	(4) ∀y(~Ray → a=y)	2 &E
5	(5) ~Pb	A
	(n-3) ∃y(y≠b & Ryb)	new goal
	(n-2) ~Pb → ∃y(y≠b & Ryb)	→I
	(n-1) ∀x(~Px → ∃y(y≠x & Ryx))	∀I
	(n) ∀x(~Px → ∃y(y≠x & Ryx))	∃E

Step 2: I note that my goal is an existential so if I could prove an instance of it I would be done. Since one of the sentences to prove would be y≠b y is obviously not going to be ‘b’. It is now clear that it needs to be ‘a’. We can easily get the a≠b part, now we just need the Rab part to have an instance of our goal.

1	(1) ∃x(Px & ∀y(~Rxy → x=y))	A
2	(2) Pa & ∀y(~Ray → a=y)	A
2	(3) Pa	2 &E
2	(4) ∀y(~Ray → a=y)	2 &E
5	(5) ~Pb	A
2,5	(6) a≠b	3,5 NI
	(n-4) a≠b & Rab	&I
	(n-3) ∃y(y≠b & Ryb)	∃I
	(n-2) ~Pb → ∃y(y≠b & Ryb)	→I
	(n-1) ∀x(~Px → ∃y(y≠x & Ryx))	∀I
	(n) ∀x(~Px → ∃y(y≠x & Ryx))	∃E

Step 3: To get Rab I notice that since I have a≠b, I can plug ‘b’ into line 4 to get something useful. It turns out to be just exactly what I need.

1	(1) ∃x(Px & ∀y(~Rxy → x=y))	A
2	(2) Pa & ∀y(~Ray → a=y)	A
2	(3) Pa	2 &E
2	(4) ∀y(~Ray → a=y)	2 &E

	5	(5) ~Pb	A
	2,5	(6) a≠b	3,5 NI
	2	(7) ~Rab → a=b	4 ∀E
	2,5	(8) Rab	6,7 MT
	2,5	(9) a≠b & Rab	6,8 &I
	2,5	(10) ∃y(y≠b & Ryb)	9 ∃I
2	(11) ~Pb → ∃y(y≠b & Ryb)		10→I(5)
2	(12) ∀x(~Px → ∃y(y≠x & Ryx))		11 ∀I
1	(13) ∀x(~Px → ∃y(y≠x & Ryx))		1,12 ∃E(2)